An Analytic Approach To Series For Differential Equations.

The differential equation, especally of second order, is indispensible to describe natural phenomena. For this reason almost all students and engineers who study natural science and/or engineering are supposed to learn this unbelievably powerful tool. But unfor-tunately only a few of these equations can be solved elemen-tarily. Thus we are not unfrequently forced to rely on the Methods of Numerical Analysis with computers at hand. There are so many good algorithms to attack differential equations with. But numerical method gives us only a specific solutions at given initial and boundary conditions. Applying Taylor Series to these differential equations, I have been trying to get analytical solutions of them. With Taylor Series, we can approximate as accurately as possible, if expressed explicitly, to the actual solution. But some differential equations are even more difficult than we expect. The following method is not entirely new but quite a lot, I think, sophisticated, if lucky, to get general solution. The readers of this paper are supposed to have moderate knowledge of differential equations.

A large class of ordinary differential equations possesses solutions expressible, over a certain interval, in terms of power series. Before investigating methods of obtaining such solutions, we review certain useful properties of power series.

An expression of the form

$$\sum_{n=0}^{\infty} a(n)(x-x_0)^n = a(0) + a(1)(x-x_0) + a(2)(x-x_0)^2 + \cdots \qquad (1)$$

is called power series and is defined as the limit

$$\sum_{n=0}^{\infty} a(n)(x-x_0)^n = \lim_{N\to\infty} \sum_{n=0}^{N} a(n)(x-x_0)^n$$

for those values of x for which the limit exists. For such values of x the series is said to converge. We will suppose that the variable x and the coefficients are real.

To determine for what values of x the series (1) converges, we may make use of the values of x the ratio test, which states that, if the absolute value of the ratio of the $(n+1)$th term to the nth term in any infinite series approaches a limit ρ as $n \to \infty$, then the series converges when $\rho < 1$ and diverges when $\rho > 1$. The test fails if $\rho = 1$. A more delicate test states that, if the absolute value of the same ratio is bounded by some number σ as $n \to \infty$, then the series converges when $\sigma < 1$. In the case of the power series (1) we obtain

$$\rho = \lim_{n\to\infty} \left| \frac{a(n+1)}{a(n)} \right| |x - x_0| = L|x - x_0|,$$

where

$$L = \lim_{n\to\infty} \left| \frac{a(n+1)}{a(n)} \right| \tag{2}$$

if the last limit exists. In this case it follows that (1) converges when

$$|x - x_0| < \frac{1}{L}$$

and diverges when

$$|x - x_0| > \frac{1}{L}.$$

Thus, when L exists and is finite, an interval of convergence

$$\left(x_0 - \frac{1}{L},\ x_0 + \frac{1}{L} \right)$$

is determined symmetrically about the point x_0, such that inside the interval the series converges and outside the interval it diverges. The distance $R = 1/L$ is frequently called the radius of convergence.

The behavior of the series at the end points of the interval is not determined by the ratio test. We will go into the analytical details of series nor disturb ourselves with the radius of convergence, but concenterate at the solution of differential equation with series.

Here I, the writer of this paper, quote some useful formulas from "Mathematical Handbook of Formulas and Table" and modified some of them for our use.

Any analytic function may be expressed in Taylor series of the form

$$f(x) = \sum_{n=0}^{\infty} a(n)\,(x-x_0)^n \tag{2}$$

where

$$a(n) = \frac{f^{(n)}(x_0)}{n!}, \quad f^{(n)}(x_0) = \frac{d^n}{dx^n} f(x)\Big|_{x=x_0},$$

when $x_0 = 0$, then

$$f(x) = \sum_{n=0}^{\infty} a(n)\, x^n \tag{3}$$

where $a(n) = \dfrac{f^{(n)}(0)}{n!}, \quad f^{(n)}(0) = \dfrac{d^n}{dx^n} f(x)\Big|_{x=0}.$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{m=0}^{\infty} \frac{x^{4m+1}}{(4m+1)!} - \sum_{m=0}^{\infty} \frac{x^{4m+3}}{(4m+3)!}, \tag{1M}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sum_{m=0}^{\infty} \frac{x^{4m}}{(4m)!} - \sum_{m=0}^{\infty} \frac{x^{4m+2}}{(4m+2)!}. \tag{2M}$$

$$B(2n) = \frac{1}{2} - \frac{1}{n+1} - (2n)! \sum_{k=0}^{n-2} \frac{B(2k+2)}{(2n-2k-1)!\,(2k+2)!}, \tag{1R}$$

where $B(2n)$ is called Bernouilli numbers, with

$B(0) = 1, \quad B(1) = -\frac{1}{2}, \quad B(2) = \frac{1}{6}$, and

$B(2n+1) = 0$, for all $n \geq 1$.

$$E(2n) = -(2n)! \sum_{k=0}^{n-1} \frac{E(2k)}{(2n-2k)!\,(2k)!} \tag{2R}$$

where $E(2m)$ is called Euler numbers, with

$E(0)=1$, and $E(2m-1)=0$ for all $m \geq 1$.

$$\tan x = \sum_{m=1}^{\infty} \frac{2^{2m}(2^{2m}-1)}{(2m)!} |B(2m)| \, x^{2m-1} \qquad x^2 < \frac{\pi^2}{4}. \qquad (3R)$$

$$\cot x = \frac{1}{x} - \sum_{m=1}^{\infty} \frac{2^{2m}|B(2m)|}{(2m)!} x^{2m-1} \qquad x^2 < \pi^2. \qquad (4R)$$

$$\sec x = \sum_{m=0}^{\infty} \frac{|E(2m)|}{(2m)!} x^{2m} \qquad x^2 < \frac{\pi^2}{4}. \qquad (5R)$$

$$\csc x = \frac{1}{x} + \sum_{m=1}^{\infty} \frac{2(2^{2m-1}-1)}{(2m)!} |B(2m)| \, x^{2m-1} \qquad x^2 < \pi^2. \qquad (6R)$$

$$\sin^2 x = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} 2^{2m-1}}{(2m)!} x^{2m} \qquad (7R)$$

$$\cos^2 x = 1 - \sum_{m=1}^{\infty} \frac{(-1)^{m+1} 2^{2m-1}}{(2m)!} x^{2m} \qquad (8R)$$

$$\sin^3 x = \frac{1}{4} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(3^{2m+1}-3)}{(2m+1)!} x^{2m+1} \qquad (9R)$$

$$\cos^3 x = \frac{1}{4} \sum_{m=0}^{\infty} \frac{(-1)^{m}(3^{2m}+3)}{(2m)!} x^{2m} \qquad (10R)$$

$$\sin^{2p} x = \frac{1}{2^{2p}} \sum_{m=0}^{p-1} (-1)^{p-m}\, 2 \binom{2p}{m} \cos 2(p-m)x + \frac{1}{2^{2p}} \binom{2p}{p}. \qquad (11R)$$

$$\sin^{2p-1} x = \frac{1}{2^{2p-2}} \sum_{m=0}^{p-1} (-1)^{p+m-1} \binom{2p-1}{m} \sin(2p-2m-1)x \qquad (12R)$$

$$\cos^{2p} x = \frac{1}{2^{2p}} \sum_{m=0}^{p-1} 2\binom{2p}{m} \cos 2(p-m)x + \frac{1}{2^{2p}} \binom{2p}{p}. \qquad (13R)$$

$$\cos^{2p-1} x = \frac{1}{2^{2p-2}} \sum_{m=0}^{p-1} \binom{2p-1}{m} \cos(2p-2m-1)x, \qquad (14R)$$

where p is a positive integer.

$$\sin^{-1} x = \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}(m!)^2(2m+1)} x^{2m+1} \qquad x^2<1 \qquad (15R)$$

$$\tan^{-1} x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)} \qquad x^2<1 \qquad (16R)$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad (17R)$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad (18R)$$

$$\tanh x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B(2n)\, x^{2n-1} \qquad x^2 < \frac{\pi^2}{4} \qquad (19R)$$

$$\coth x = \sum_{n=1}^{\infty} \frac{2^{2n} B(2n)}{(2n)!} x^{2n-1} \qquad x^2 < \pi^2 \qquad (20R)$$

$$\operatorname{sech} x = \sum_{n=0}^{\infty} \frac{E(2n)}{(2n)!} x^{2n} \qquad x^2 < \frac{\pi^2}{4} \qquad (21R)$$

$$\cosh x = \frac{1}{x} - \sum_{n=0}^{\infty} \frac{2(2^{2n-1}-1)}{(2n)!} B(2n)\, x^{2n-1} \qquad x^2 < \pi^2 \qquad (22R)$$

$$\sinh^{2p} x = \frac{(-1)^p}{2^{2p}} \sum_{n=0}^{p-1} (-1)^{p-n}\, 2 \binom{2p}{n} \cosh 2(p-n)x + \frac{(-1)^p}{2^{2p}} \binom{2p}{p} \qquad (23R)$$

$$\sinh^{2p-1} x = \frac{(-1)^{n-1}}{2^{2p-2}} \sum_{n=0}^{p-1} (-1)^{p+n-1} \binom{2p-1}{n} \sinh (2p-2n-1)x \qquad (24R)$$

$$\cosh^{2p} x = \frac{1}{2^{2p}} \sum_{n=0}^{p-1} 2 \binom{2p}{n} \cosh 2(p-n)x + \frac{1}{2^{2p}} \binom{2p}{p} \qquad (25R)$$

$$\cosh^{2p-1} x = \frac{1}{2^{2p-2}} \sum_{n=0}^{p-1} \binom{2p-1}{n} \cosh (2p-2n-1)x \qquad (26R)$$

$$\sinh^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!\, x^{2n+1}}{2^{2n} (n!)^2 (2n+1)} \qquad x^2 < 1 \qquad (27R)$$

$$\tanh^{-1} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)} \qquad x^2 < 1 \qquad (28R)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad (4)$$

$$a^x = \sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} x^n \qquad (29R)$$

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \qquad (30R)$$

$$e^x (1+x) = \sum_{n=0}^{\infty} \frac{(n+1)}{n!} x^n \qquad (31R)$$

$$(1+x)^p = \sum_{n=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(n+1)\Gamma(p-n+1)} x^n \qquad (5)$$

where $p \neq 0, \pm 1, \pm 2, \cdots,$

$$(1+x)^p = \sum_{n=0}^{p} \frac{\Gamma(p+1)}{\Gamma(n+1)\Gamma(p-n+1)} x^n , \tag{6}$$

where $p = 1, 2, 3, \cdots,$

$$(1+x)^{-p} = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+p)}{\Gamma(n+1)\ \Gamma(p)} x^n \tag{7}$$

where $p = 1, 2, 3, \cdots.$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{32R}$$

$$= \sum_{n=0}^{\infty} x^{2n} + \sum_{n=0}^{\infty} x^{2n+1} \qquad -1 < x < 1 \tag{8}$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \qquad 0 < x \leq 2 \tag{33R}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} x^{n+1} \qquad -1 < x \leq 1 \tag{9}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} - \sum_{n=0}^{\infty} \frac{x^{2n+2}}{(2n+2)} \tag{10}$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)} x^{2n-1} \qquad x^2 < 1 \tag{34R}$$

$$\ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad -1 \leq x < 1 \tag{35R}$$

Now I will state here some series operations and derive some of them. Most of the following formulas are derived by me.

Let $f(x)$ be analytic at $x=0$, then

$$f(x) = \sum_{n=0}^{\infty} a(n)\, x^n \tag{11}$$

which is equivalent to

$$f(x) = \sum_{n=0}^{\infty} a(2n)\, x^{2n} + \sum_{n=0}^{\infty} a(2n+1)\, x^{2n+1} \qquad (11)$$

Proof) Let $\sum_{n=0}^{\infty} a(n) x^n$ be a uniformly convergent power series. Then,

$$\sum_{n=0}^{\infty} a(n)\, x^n = a(0) + a(1)x + a(2)x^2 + \cdots + a(n)x^n + \cdots,$$
$$= a(0) + a(2)x^2 + a(4)x^4 + \cdots + a(2n)x^{2n} + \cdots,$$
$$+ a(1)x + a(3)x^3 + \cdots + a(2n+1)x^{2n+1} + \cdots$$
$$= \sum_{n=0}^{\infty} a(2n)\, x^{2n} + \sum_{n=0}^{\infty} a(2n+1)\, x^{2n+1}$$

is also a uniformly convergent power series.

Example)

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(ix)^{2n} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(i^2)^n x^{2n} + \sum_{n=0}^{\infty} \frac{i(i^2)^n\, x^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!}$$
$$= \cos x + i \sin x.$$

Shifting indices 1 :

$$\sum_{n=\alpha}^{\infty} a(n)\, x^n = \sum_{n=0}^{\infty} a(n+\alpha)\, x^{n+\alpha} \qquad (12)$$

Proof)

$$\sum_{n=\alpha}^{\infty} a(n)\, x^n = a(\alpha)\, x^{\alpha} + a(\alpha+1)\, x^{\alpha+1} + \cdots$$
$$= a(0+\alpha)\, x^{0+\alpha} + a(1+\alpha)\, x^{1+\alpha} + a(2+\alpha)\, x^{2+\alpha} + \cdots$$
$$= \sum_{n=0}^{\infty} a(n+\alpha)\, x^{n+\alpha}$$

First order differentiation :

$$f(x) = \sum_{n=0}^{\infty} a(n)\, x^n, \quad \text{then}$$

$$f'(x) = \sum_{n=1}^{\infty} n\, a(n)\, x^{n-1}, \tag{13}$$

which is equivalent to

$$f'(x) = \sum_{n=0}^{\infty} (n+1)\, a(n+1)\, x^n, \quad \text{and to} \tag{14}$$

$$= \sum_{n=1}^{\infty} (2n)\, a(2n)\, x^{2n-1} + \sum_{n=0}^{\infty} (2n+1)\, a(2n+1)\, x^{2n} \tag{15}$$

$$= \sum_{n=0}^{\infty} (2n+1)\, a(2n+1)\, x^{2n} + \sum_{n=0}^{\infty} (2n+2)\, a(2n+2)\, x^{2n+1} \tag{16}$$

Proof)

$$f(x) = \sum_{n=0}^{\infty} a(n)\, x^n$$

$$= a(0) + a(1)x + a(2)x^2 + \cdots + a(n)x^n + \cdots$$

$$f'(x) = a(1) + 2a(2)x + \cdots + n\, a(n)\, x^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} n\, a(n)\, x^{n-1},$$

by equation (12) we have

$$f'(x) = \sum_{n=0}^{\infty} (n+1)\, a(n+1)\, x^n.$$

By equation (1C) we have

$$f(x) = a(0) + a(2)x^2 + a(4)x^4 + \cdots + a(2n)x^{2n} + \cdots$$
$$+ a(1)x + a(3)x^3 + a(5)x^5 + \cdots + a(2n+1)x^{2n+1} + \cdots$$

$$f'(x) = 2a(2)x + 4a(4)x^3 + \cdots + (2n)\, a(2n)\, x^{2n-1} + \cdots$$
$$+ a(1) + 3a(3)x^2 + 5a(5)x^4 + \cdots + (2n+1)\, a(2n+1)\, x^{2n} + \cdots$$
$$= \sum_{n=1}^{\infty} (2n)\, a(2n)\, x^{2n-1} + \sum_{n=0}^{\infty} (2n+1)\, a(2n+1)\, x^{2n},$$

using equation (12) we can have

$$f'(x) = \sum_{n=0}^{\infty} (2n+1)\, a(2n+1)\, x^{2n} + \sum_{n=0}^{\infty} (2n+2)\, a(2n+2)\, x^{2n+1},$$

thus we proved equation (16).

2nd order differentiation:

$$f'(x) = \sum_{n=1}^{\infty} n\,a(n)\,x^{n-1}, \quad \text{then}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)\,a(n)\,x^{n-2} \tag{17}$$

which is equivalent to

$$f''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)\,a(n+2)\,x^{n}, \tag{18}$$

$$f''(x) = \sum_{m=1}^{\infty} (2m)(2m-1)\,a(2m)\,x^{2m-2} + \sum_{m=1}^{\infty} (2m+1)(2m)\,a(2m+1)\,x^{2m-1}, \tag{19}$$

and to

$$f''(x) = \sum_{m=0}^{\infty} (2m+2)(2m+1)\,a(2m+2)\,x^{2m} + \sum_{m=0}^{\infty} (2m+3)(2m+2)\,a(2m+3)\,x^{2m+1}. \tag{20}$$

Proof)

Since

$$f'(x) = \sum_{n=1}^{\infty} n\,a(n)\,x^{n-1},$$

$$= a(1) + 2a(2)x + 3a(3)x^2 + \cdots + n\,a(n)\,x^{n-1} + \cdots,$$

$$f''(x) = 2a(2) + 3\cdot 2\,a(3)x + \cdots + n(n-1)\,a(n)\,x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)\,a(n)\,x^{n-2},$$

thus equation (17) is proved.

Equation (15)

$$f'(x) = \sum_{m=1}^{\infty} (2m)\,a(2m)\,x^{2m-1} + \sum_{m=0}^{\infty} (2m+1)\,a(2m+1)\,x^{2m}$$

$$= 2a(2)x + 4a(4)x^3 + 6a(6)x^5 + \cdots + (2m)\,a(2m)\,x^{2m-1} + \cdots$$
$$+ a(1)x + 3a(3)x^2 + 5a(5)x^4 + \cdots + (2m+1)\,a(2m+1)\,x^{2m} + \cdots.$$

Taking derivative

$$f''(x) = 2a(2) + 4\cdot 3\,a(4)x^2 + \cdots + (2m)(2m-1)\,a(2m)\,x^{2m-2} + \cdots$$
$$+ a(1) + 3\cdot 2\,a(3)x + \cdots + (2m+1)(2m)\,a(2m+1)\,x^{2m-1} + \cdots$$

$$f''(x) = \sum_{m=1}^{\infty} (2m)(2m-1)\,a(2m)\,x^{2m-2} + \sum_{m=1}^{\infty} (2m+1)(2m)\,a(2m+1)\,x^{2m-1}$$

thus equation (19) is proved.

By equation (12)

$$f''(x) = \sum_{n=0}^{\infty} (2(n+1))\ (2(n+1)-1)\ a(2(n+1))\ x^{2(n+1)-2}$$

$$+ \sum_{n=0}^{\infty} (2(n+1)+1)\ (2(n+1))\ a(2(n+1)+1)\ x^{2(n+1)-2}$$

$$= \sum_{m=0}^{\infty} (2m+2)\ (2m+1)\ a(2m+2)\ x^{2m} + \sum_{n=0}^{\infty} (2m+3)\ (2m+2) a(2m+3) x^{2m+1},$$

therefore we proved equation (20).

Shifting indices 2 :

$$\sum_{n=0}^{\infty} a(n)\, x^{n+\beta} = \sum_{m=\beta}^{\infty} a(n-\beta)\, x^{n} \tag{21}$$

Proof) $\sum_{n=0}^{\infty} a(n)\, x^{n+\beta} = a(0)\, x^{\beta} + a(1)\, x^{\beta+1} + a(2)\, x^{\beta+2} + \cdots$

$$= \sum_{n=\beta}^{\infty} a(n-\beta)\, x^{n}.$$

Term by term addition of two convergent power series :

$$\sum_{n=0}^{\infty} a(n) x^n + \sum_{n=0}^{\infty} b(n) x^n = \sum_{n=0}^{\infty} [a(n) + b(n)]\, x^n. \tag{22}$$

Multiplication of two convergent power series :

$$\sum_{n=0}^{\infty} a(n)\, x^n \times \sum_{n=0}^{\infty} b(n) x^n$$

$$= \sum_{n=0}^{\infty} b(n)\, x^n \times \sum_{n=0}^{\infty} a(n)\, x^n$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a(k)\, b(n-k) \right] x^n$$

$$= \sum_{m=0}^{\infty} \left[\sum_{k=0}^{n} b(k)\, a(n-k) \right] x^n \tag{23}$$

Proof) $\sum_{n=0}^{\infty} a(n)\, x^n \times \sum_{n=0}^{\infty} b(n)\, x^n$

$$= \sum_{k=0}^{0} a(k)\, b(0-k) + \sum_{k=0}^{1} a(k)\, b(1-k)\, x + \sum_{k=0}^{2} a(k)\, b(2-k)\, x^2$$

$$+ \cdots + \sum_{k=0}^{n} a(k)\, b(m-k)\, x^n + \cdots$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a(k) b(n-k) \right] x^n = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} b(k) a(n-k) \right] x^n .$$

$$\sum_{n=0}^{\infty} a(n) x^{2n} \times \sum_{n=0}^{\infty} b(n) x^{2n} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} a(k) b(n-k) \right] x^{2n} \quad (24)$$

$$\sum_{n=0}^{\infty} a(n) x^{2n} \times \sum_{n=0}^{\infty} b(n) x^{2n+1} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} a(k) b(n-k) \right] x^{2n+1} \quad (25)$$

$$\sum_{n=0}^{\infty} a(n) x^{2n+1} \times \sum_{n=0}^{\infty} b(n) x^{2n+1} = \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n-1} a(k) b(n-k-1) \right] x^{2n} \quad (26)$$

Proof) $\sum_{n=0}^{\infty} a(n) x^{2n} \times \sum_{n=0}^{\infty} b(n) x^{2n}$

$$= \sum_{k=0}^{0} a(k) b(0-k) + \sum_{k=0}^{1} a(k) b(1-k) x^2$$

$$+ \cdots + \sum_{k=0}^{n} a(k) b(n-k) x^{2n} + \cdots$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a(k) b(n-k) \right] x^{2n}$$

$$\sum_{n=0}^{\infty} a(n) x^{2n} \times \sum_{n=0}^{\infty} b(n) x^{2n+1}$$

$$= x \sum_{n=0}^{\infty} a(n) x^{2n} \times \sum_{n=0}^{\infty} b(n) x^{2n}$$

$$= x \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a(k) b(n-k) \right] x^{2n}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a(k) b(n-k) \right] x^{2n+1}$$

$$\sum_{n=0}^{\infty} a(n) x^{2n+1} \times \sum_{n=0}^{\infty} b(n) x^{2n+1}$$

$$= x^2 \sum_{n=0}^{\infty} a(n) x^{2n} \times \sum_{n=0}^{\infty} b(n) x^{2n}$$

$$= x^2 \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a(k) b(n-k) \right] x^{2n}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a(k) b(n-k) \right] x^{2n+2}$$

$$= \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n-1} a(k) b(n-k) \right] x^{2n} .$$

Shifting indices 3:

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} a(k)\, b(n-k)\right] x^{n+\alpha} = \sum_{n=\alpha}^{\infty}\left[\sum_{k=0}^{n-\alpha} a(k)\, b(n-k-\alpha)\right] x^{n} \quad (27)$$

Example)

(1) $$\sum_{n=1}^{\infty} 2^n x^{n+1} + \sum_{n=0}^{\infty} (1+n)\, x^n$$

$$= \sum_{n=2}^{\infty} 2^{n-1} x^n + 1 + 2x + \sum_{n=2}^{\infty} (1+n)\, x^n$$

$$= 1 + 2x + \sum_{n=0}^{\infty} \left(2^{n-1} + 1 + n\right) x^n .$$

(2) $$\sum_{n=2}^{\infty} x^{n-1} + \sum_{n=3}^{\infty} n!\, x^{n+2}$$

$$= \sum_{n=1}^{\infty} x^n + \sum_{n=5}^{\infty} (n-2)!\, x^n$$

$$= \sum_{n=1}^{4} x^n + \sum_{n=5}^{\infty} x^n + \sum_{n=5}^{\infty} (n-2)!\, x^n$$

$$= \sum_{n=1}^{4} x^n + \sum_{n=5}^{\infty} \left(1 + (n-2)!\right) x^n$$

$$= x + x^2 + x^3 + x^4 + \sum_{n=5}^{\infty} \left(1 + (n-2)!\right) x^n .$$

I state 6 more equations that will be used.

$$\sum_{n=0}^{2\alpha} a(n) = \sum_{n=0}^{\alpha} a(2n) + \sum_{n=0}^{\alpha-1} a(2n+1) \qquad \alpha = 1, 2, 3, \dots \quad (28)$$

$$\sum_{n=0}^{2\alpha+1} a(n) = \sum_{n=0}^{\alpha} a(2n) + \sum_{n=0}^{\alpha} a(2n+1) \qquad \alpha = 0, 1, 2, \dots \quad (29)$$

$$\sum_{n=0}^{2\alpha-1} a(n) = \sum_{n=0}^{\alpha-1} a(2n) + \sum_{n=0}^{\alpha-1} a(2n+1) \qquad \alpha = 1, 2, 3, \dots \quad (30)$$

$$\sum_{n=1}^{2\alpha} a(n) = \sum_{n=1}^{\alpha} a(2n-1) + \sum_{n=1}^{\alpha} a(2n) \qquad \alpha = 1, 2, 3, \dots \quad (31)$$

$$\sum_{n=1}^{2\alpha-1} a(n) = \sum_{n=1}^{\alpha} a(2n-1) + \sum_{n=1}^{\alpha-1} a(2n) \qquad \alpha = 1, 2, 3, \dots \quad (32)$$

$$\sum_{n=1}^{2\alpha+1} a(n) = \sum_{n=1}^{\alpha+1} a(2n-1) + \sum_{n=1}^{\alpha} a(2n) \qquad \alpha = 1, 2, 3, \dots \quad (33)$$

Power Series Solutions of Differential Equations

The power series method consists of substituting $y = \sum_{n=0}^{\infty} a(n)x^n$ into the differential equation and attempting to solve for $a(n)$'s to obtain a solution. Success is not guaranted under all circumstances, but the following theorem gives us cause for optimism under fairly general conditions.

1st order case :

If $g(x)$ and $r(x)$ are analytic on an interval I, then every solution of

$$y' + g(x)y = r(x) \tag{34}$$

is also analytic on the interval I and can hence be expressed as a power series

$$y = \sum_{n=0}^{\infty} a(n)x^n$$

2nd order case :

If $p(x)$, $g(x)$, and $f(x)$ are analytic on an interval I, then every solution of

$$y'' + p(x)y' + g(x)y = f(x) \tag{35}$$

is also analytic on the interval I and can be expressed as a power series

$$y = \sum_{n=0}^{\infty} a(n)x^n$$

Note : If $r(x)=0$ and $y(x=0) \neq 0$, then the homogeneous solution of equation (34) is

$$y = a(0)\sum_{n=0}^{\infty} \langle n \rangle x^n$$

where

$$a(n) = a(0)\langle n \rangle, \quad \langle 0 \rangle = 1.$$

Example) Solve the following differential equations.

4) $y' + G(x)\,y = 0$ (36)

Let $G(x)$ be analytic on an interval I such that $G(x)$ can be expressed as follows;

$$G(x) = \sum_{n=0}^{\infty} g(n)\,x^n . \qquad (37)$$

Then substituting equation (37) into equation (36), we have

$$y' + \left[\sum_{n=0}^{\infty} g(n)\,x^n\right] y = 0 . \qquad (38)$$

Now let $y = \sum_{n=0}^{\infty} a(n)\,x^n$, then by equation (4), we have

$$y' = \sum_{n=0}^{\infty} (n+1)\,a(n+1)\,x^n . \qquad (4)$$

Substituting the above two equations into equation (38), we have

$$\sum_{n=0}^{\infty} (n+1)\,a(n+1)\,x^n + \left[\sum_{n=0}^{\infty} g(n)\,x^n\right] \times \left[\sum_{n=0}^{\infty} a(n)\,x^n\right] = 0 \qquad (39)$$

By equation (22) we have

$$\sum_{n=0}^{\infty} (n+1)\,a(n+1)\,x^n + \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} a(k)\,g(n-k)\right] x^n = 0 \qquad (40)$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+1)\,a(n+1) + \sum_{k=0}^{n} a(k)\,g(n-k)\right] x^n = 0 .$$

We let

$$(n+1)\,a(n+1) + \sum_{k=0}^{n} a(k)\,g(n-k) = 0 \quad \text{for } n = 0, 1, 2, \cdots$$

$$\Rightarrow a(n+1) = -\frac{\sum_{k=0}^{n} a(k)\,g(n-k)}{(n+1)} . \qquad (41)$$

Letting $a(n+1) = a(0)\langle n+1\rangle$ and $a(k) = a(0)\langle k\rangle$ with $\langle 1\rangle = 1$, we have

$$a\langle 0\rangle \langle n+1\rangle = -\frac{a(0)\sum_{k=0}^{n} \langle k\rangle\, g(n-k)}{(n+1)}$$

$$\Rightarrow \langle n+1\rangle = -\frac{\sum_{k=0}^{n} \langle k\rangle\, g(n-k)}{(n+1)} \quad \text{for } n = 0, 1, 2, \cdots$$

shifting indices,

$$\langle n \rangle = -\frac{\sum_{k=0}^{n-1} \langle k \rangle g(n-k-1)}{n} \qquad \text{for } n = 1, 2, 3, \cdots. \quad (42)$$

Since $y = \sum_{n=0}^{\infty} a(n) x^n$, we thus have the solution of equation (36) as follows;

$$y = a(0) \sum_{n=0}^{\infty} \langle n \rangle x^n \qquad (43)^*$$

$$= a(0) + a(0) \sum_{n=1}^{\infty} \langle n \rangle x^n$$

where $\langle 0 \rangle = 1$, $\langle n \rangle = -\frac{\sum_{k=0}^{n-1} \langle k \rangle g(n-k-1)}{n}$ for $n = 1, 2, 3, \cdots$ (43)

and letting $y(x=0) = y_0$, we can determine $a(0)$.

$$y(x=0) = y_0 = a(0) + a(0) \sum_{n=1}^{\infty} \langle n \rangle (0)^n,$$

thus we have

$$a(0) = y_0.$$

In summary,

$$y' + G(x) y = 0 \qquad (36)$$

has solution

$$y = y_0 + y_0 \sum_{n=1}^{\infty} \langle n \rangle x^n, \qquad (44)$$

where $\langle 0 \rangle = 1$, $\langle n \rangle = -\frac{\sum_{k=0}^{n-1} \langle k \rangle g(n-k-1)}{n}$ for $n = 1, 2, 3, \ldots$,

and $G(x) = \sum_{n=0}^{\infty} g(n) x^n$.

If $G(x) = e^x$, then by equation (4)

$$G(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ thus } g(n) = \frac{1}{n!},$$

$$g(n-k-1) = \frac{1}{(n-k-1)!}. \qquad (45)$$

Substituting equation (45) into equation (43), we have

$$\langle n\rangle = -\frac{\sum_{k=0}^{n-1} \frac{1}{(n-k-1)!}}{n} \quad \text{with } \langle 0\rangle = 1, \ n=1,2,3,\cdots. \tag{46}$$

Since $y(x=0) = e^0 = 1$,

$y' + e^x y = 0$ has the solution

$$y = 1 + \sum_{n=1}^{\infty} \langle n\rangle x^n$$

where

$$\langle n\rangle = -\frac{1}{n}\left[\sum_{k=0}^{n-1} \frac{\langle k\rangle}{(n-k-1)!}\right], \ \langle 0\rangle = 1. \tag{47}$$

This problem will be considered later again to expand $\langle n\rangle$'s with the aid of computer.

Now I state another useful multiplication formula,

$$\sum_{n=0}^{\alpha} g(n) x^n \times \sum_{n=0}^{\infty} a(n) x^n$$

$$= \sum_{n=0}^{\alpha}\left[\sum_{k=0}^{n} g(k)a(n-k)\right]x^n + \sum_{n=\alpha+1}^{\infty}\left[\sum_{k=0}^{\alpha} g(k)a(n-k)\right]x^n \tag{48}$$

Proof) Let $\sum_{n=0}^{\alpha} g(n)x^n = \sum_{n=0}^{\infty} g(n)x^n$, where we assume

$$g(n) = 0 \text{ for all } n \geq \alpha+1. \tag{49}$$

Then by equation (23),

$$\sum_{n=0}^{\infty} g(n)x^n \times \sum_{n=0}^{\infty} a(n)x^n$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} g(k)a(n-k)\right]x^n$$

$$= \sum_{n=0}^{\alpha}\left[\sum_{k=0}^{n} g(k)a(n-k)\right]x^n + \sum_{n=\alpha+1}^{\infty}\left[\sum_{k=0}^{n} g(k)a(n-k)\right]x^n$$

$$= \sum_{n=0}^{\alpha}\left[\sum_{k=0}^{n} g(k)a(n-k)\right]x^n$$

$$+ \sum_{n=\alpha+1}^{\infty}\left[\sum_{k=0}^{\alpha} g(k)a(n-k) + \sum_{k=\alpha+1}^{n} g(k)a(n-k)\right]x^n. \tag{50}$$

Since $g(n) = 0$ for all $n \geq \alpha+1$, $\sum_{k=\alpha+1}^{n} g(k)a(n-k) = 0$,

thus we have

$$\sum_{n=0}^{\infty} g(n)x^n \times \sum_{n=0}^{\infty} a(n)x^n$$
$$= \sum_{n=0}^{\alpha}\left[\sum_{k=0}^{n} g(k)a(n-k)\right]x^n + \sum_{n=\alpha+1}^{\infty}\left[\sum_{k=0}^{\alpha} g(k)a(n-k)\right]x^n .$$

(2) Let $G(x)$ be a polynomial function such that $G(x) = \sum_{n=0}^{\alpha} g(n)x^n$, $g(n) \neq 0$, and $G(x)$ is analytic at $x=0$.

Then we are going to solve

$$y' + G(x)y = 0 \tag{51}$$

Substituting $y = \sum_{n=0}^{\infty} a(n)x^n$ into equation (51), we have

$$\sum_{n=0}^{\infty}(n+1)a(n+1)x^n + \sum_{n=0}^{\alpha} g(n)x^n \times \sum_{n=0}^{\infty} a(n)x^n = 0. \tag{52}$$

By equation (48) we have

$$\sum_{n=0}^{\infty}(n+1)a(n+1)x^n + \sum_{n=0}^{\alpha}\left[\sum_{k=0}^{n} g(k)a(n-k)\right]x^n + \sum_{n=\alpha+1}^{\infty}\left[\sum_{k=0}^{\alpha} g(k)a(n-k)\right]x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\alpha}(n+1)a(n+1)x^n + \sum_{n=0}^{\alpha}\left[\sum_{k=0}^{n} g(k)a(n-k)\right]x^n$$
$$+\sum_{n=\alpha+1}^{\infty}(n+1)a(n+1)x^n + \sum_{n=\alpha+1}^{\infty}\left[\sum_{k=0}^{\alpha} g(k)a(n-k)\right]x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\alpha}\left[(n+1)a(n+1) + \sum_{k=0}^{n} g(k)a(n-k)\right]x^n$$
$$+\sum_{n=\alpha+1}^{\infty}\left[(n+1)a(n+1) + \sum_{k=0}^{\alpha} g(k)a(n-k)\right]x^n = 0 .$$

Thus we have,

$$a(n+1) = -\frac{\sum_{k=0}^{n} g(k)a(n-k)}{(n+1)}, \quad n = 0, 1, 2, \cdots, \alpha. \tag{53}$$

and

$$a(n+1) = -\frac{\sum_{k=0}^{\alpha} g(k)a(n-k)}{(n+1)}, \quad n = \alpha+1, \alpha+2, \cdots \tag{54}$$

where $\alpha = 0, 1, 2, \cdots$

If $\alpha = 0$ then $G(x) = \beta$ (constant), then substiting $g(k) = \beta$ and putting $n=0$ in equation (53), we have

$$a(1) = -\beta a(0)$$
$$\Rightarrow a(0)\langle 1\rangle = -\beta a(0)\langle 0\rangle, \text{ where } \langle 0\rangle = 1,$$

therefore $\langle 1\rangle = -\beta$. (55)

Now putting $\alpha = 0$ and substituting $g(x) = \beta$ in equation (54) we have

$$a(n+1) = -\frac{\beta}{(n+1)}a(n),$$

$$\rightarrow a(0)\langle n+1\rangle = -\frac{\beta}{(n+1)}a(0)\langle n\rangle,$$

thus we have,

$$\langle n+1\rangle = -\frac{\beta}{(n+1)}\langle n\rangle \qquad n = 1, 2, 3, \cdots$$

We can write the above equation again,

$$\langle n\rangle = -\frac{\beta}{n}\langle n-1\rangle \qquad n = 2, 3, 4, \cdots \tag{56}$$

When we expand the above recurrence relation,

$$\langle n\rangle = -\frac{\beta}{n}\left(-\frac{\beta}{n-1}\right)\left(-\frac{\beta}{n-2}\right)\cdots\left(-\frac{\beta}{2}\right)\langle 1\rangle. \tag{57}$$

Substituting equation (51) into equation (57) we have

$$\langle n\rangle = (-1)^n\frac{\beta^n}{n!} \qquad n = 2, 3, 4, \cdots$$

Thus the solution of equation (51) with $G(x) = \beta$ (constant) is

$$y = a(0) + a(1)\langle 1\rangle x + \sum_{n=2}^{\infty} a(0)\langle n\rangle x^n$$

$$= a(0)\left[1 + \langle 1\rangle x + \sum_{n=2}^{\infty}\langle n\rangle x^n\right]$$

$$= a(0)\left[1 - \beta x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\beta^n x^n\right]$$

$$= a(0)\sum_{n=0}^{\infty}\frac{1}{n!}(-\beta x)^n$$

$$= a(0)\, e^{-\beta x}$$

Since $a(0) = y_0$, $y = y_0 e^{-\beta x}$. (58)

Note: Of course the direct integration of equation (51) gives us the solution (58), but what if the direct integration of equation (51) is impossible? The above example is just to show how our series operation does work.

To obtain solution of Equ. (47) for the general case where α is any positive integer, we rewrite Equ. (49) and Equ. (50) such as

$$a(0)\langle n+1\rangle = -\frac{\sum_{k=0}^{n} g(k)\, a(0)\langle n-k\rangle}{(n+1)} \qquad n=0,1,2,\ldots,a. \quad (56)$$

leading to

$$\langle n+1\rangle = -\frac{\sum_{k=0}^{n} g(k)\langle n-k\rangle}{(n+1)} \qquad n=0,1,2,\ldots,\alpha, \quad (57)$$

shifting indices,

$$\langle n\rangle = -\frac{\sum_{k=0}^{n-1} g(k)\langle n-k-1\rangle}{n} \qquad n=1,2,3,\ldots,\alpha+1 \quad (58)$$

and

$$a(0)\langle n+1\rangle = -\frac{\sum_{k=0}^{\alpha} g(k)\, a(0)\langle n-k\rangle}{(n+1)} \qquad n=\alpha+1,\alpha+2,\ldots, \quad (59)$$

leading to

$$\langle n+1\rangle = -\frac{\sum_{k=0}^{\alpha} g(k)\langle n-k\rangle}{(n+1)} \qquad n=\alpha+1,\alpha+2,\ldots, \quad (60)$$

shifting indices,

$$\langle n\rangle = -\frac{\sum_{k=0}^{\alpha} g(k)\langle n-k-1\rangle}{n} \qquad n=\alpha+2,\alpha+3,\ldots. \quad (61)$$

Thus the solution of Equ. (47) for the general case, where α is any positive integer, i.e.

$$y' + \left(\sum_{n=0}^{\alpha} g(n)x^n\right)y = 0,$$

is

$$y = a(0) + \sum_{n=1}^{\alpha+1} a(n)\, x^n + \sum_{n=\alpha+2}^{\infty} a(n)\, x^n \qquad (62)$$

$$= a(0) + a(0)\sum_{n=1}^{\alpha+1}\langle n\rangle x^n + a(0)\sum_{n=\alpha+2}^{\infty}\langle n\rangle x^n \qquad (63)$$

where $\langle n\rangle = -\dfrac{\sum_{k=0}^{n-1} g(k)\langle n-k-1\rangle}{n}$, when $n=1,2,3,\ldots,\alpha+1$, (64) and $\langle n\rangle = -\dfrac{\sum_{k=0}^{\alpha} g(k)\langle n-k-1\rangle}{n}$, when $n = \alpha+2, \alpha+3, \ldots,$ (65)

with $\langle 0\rangle = 1$. 0

Now I derive Equ. (63), (64), and (65), directly from Equ (43)*, and from Equ.(43), to show that these equations are all equivalent.

From Equ. (43)*,

$$y = a(0) + a(0)\sum_{n=1}^{\infty}\langle n\rangle x^n,$$

then by shifting indices,

$$y = a(0) + a(0)\sum_{n=1}^{\alpha+1}\langle n\rangle x^n + a(0)\sum_{n=\alpha+2}^{\infty}\langle n\rangle x^n,$$

thus we obtain Equ. (63),

then from Equ. (43),

$$\langle n\rangle = -\frac{\sum_{k=0}^{n-1}\langle k\rangle g(n-k-1)}{n}, \quad \langle 0\rangle = 1, \quad n = 1, 2, 3, \ldots,$$

$$= -\frac{\sum_{k=0}^{n-1} g(k)\langle n-k-1\rangle}{n}, \quad n = 1, 2, 3, \ldots, \tag{66}$$

If $n \leq \alpha+1$, Equ. (66) holds, but when $n \geq \alpha+2, \ldots,$

$$\langle n\rangle = -\frac{\sum_{k=0}^{\alpha} g(k)\langle n-k-1\rangle + \sum_{k=\alpha+1}^{n-1} g(k)\langle n-k-1\rangle}{n}$$

$$= -\frac{\sum_{k=0}^{\alpha} g(k)\langle n-k-1\rangle}{n}, \quad n = \alpha+2, \alpha+3, \ldots,$$

because $g(k) = 0$, for all $k = \alpha+1, \alpha+2, \ldots$.

Therefore we proved the validity of Equ. (63).

Example) Solve $y' + b^x y = 0$. where $b \geq 0$ and (67)

$b \neq 0, 1$. By Equ. (29R), we have,

$$b^x = \sum_{n=0}^{\infty}\frac{(\ln b)^n}{(n!)}x^n, \tag{68}$$

$$\text{then, } f(n) = \frac{(\ln b)^n}{(n!)} \qquad (69)$$

Substituting Equ. (69) in Equ. (43)

$$\langle n\rangle = -\frac{\sum_{k=0}^{n-1} \langle k\rangle \frac{(\ln b)^{n-k-1}}{(n-k-1)!}}{n}, \quad \langle 0\rangle = 1, \; n=1,2,3,\dots \qquad (70)$$

$$y = a(0) + a(0)\sum_{n=1}^{\infty} \langle n\rangle x^n, \qquad (71)$$

$$\text{where } \langle n\rangle = -\frac{1}{n}\sum_{k=0}^{n-1} \langle k\rangle \frac{(\ln b)^{n-k-1}}{(n-k-1)!}, \quad \langle 0\rangle = 1, \; n=1,2,3\dots \qquad (72)$$

When $b = \exp[1] = e^1$, then

$$\langle n\rangle = -\frac{1}{n}\sum_{k=0}^{n-1} \langle k\rangle \frac{1}{(n-k-1)!}, \quad \langle 0\rangle = 1 \qquad (73)$$

This result is the same as Equ. (45)* as it should be.

When I expand $\langle n\rangle$ with computer, with $b = e^1$,

$\langle 1\rangle = -1$, $\langle 2\rangle = 0$, $\langle 3\rangle = 1.666666667E-01$

$\langle 4\rangle = 4.166666661E-02$ $\langle 5\rangle = -1.666666668E-02$

$\langle 5\rangle = -1.24999999E-02$ . . . . . .

$$y = a(0)\left(\langle 0\rangle + \langle 1\rangle x + \langle 2\rangle x^2 + \langle 3\rangle x^3 + \cdots\right) \qquad (74)$$

$$= a(0)\left(1 - x + 0.1\dot{6}x^3 + 0.041\dot{6}x^4 - 0.01\dot{6}x^5 + \cdots\right)$$

When we expand Equ. (73) with hand calculation, with $b = e^1$, we have,

$\langle 1\rangle = -1$, $\langle 2\rangle = 0$, $\langle 3\rangle = \frac{1}{6}$, $\langle 4\rangle = \frac{1}{24}\dots$

$$y = a(0)\left(1 - x + \frac{1}{6}x^3 + \frac{1}{24}x^4 \dots\right) \qquad (75)$$

As it should be Equ. (75) is same to Equ. (74).

Note: You may ask what advantages does my solution have over conventional series solution or over the conventional numerical approach? Obiously my approach does not need to know $y(x=0)$, and Solution $(43)^*$ is not a particular solution but a general solution to Equ. (37). Thus whatever function $g(x)$ may be, once it is expanded in Taylor series, its solution is ready at hand. As it were, let us suppose that we are going to solve,

$$y' + 2^x y = 0.$$

Since $g(n) = \frac{(\ln 2)^n}{n!}$, i.e. $b=2$ in Equ. (67),

we have
$$\langle n \rangle = -\frac{1}{n} \sum_{k=0}^{n-1} \langle k \rangle \frac{(\ln 2)^{n-k-1}}{(n-k-1)!}, \quad (76)$$

and expanding with computer (hand calculation is almost impossible for large n)

$\langle 1 \rangle = -1$, $\langle 2 \rangle = -0.5$, $\langle 3 \rangle = 1.666666666E-01$,

$\langle 4 \rangle = 3.75E-01$ $\langle 5 \rangle = 1.91666666E-01$

$\langle 6 \rangle = -3.47222218E-02$, $\langle 7 \rangle = -7.65625E2-02$ ...,

Therefore we have

$$y = a(0)\left[1 - x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + 0.375x^3 + 0.19\dot{6}x^4 \ldots\right].$$

Computer programme for Equ. $(43)^*$ in BASIC is in the Appendix.

Now let $r(x) \neq 0$ be a function such that $r(x)$ is analytic at $x=0$, there there exists associated Taylor series, i.e.

$$r(x) = \sum_{n=0}^{\infty} r(n) x^n, \quad (77).$$

Substituting Equ. (77) and Equ. $(37)^*$ in Equ. (35), we have,

$$y' + \left(\sum_{n=0}^{\infty} g(n) x^n\right) y = \sum_{n=0}^{\infty} r(n) x^n. \qquad (78)$$

Substituting $y = \sum_{n=0}^{\infty} a(n) x^n$ in Equ. (78), we have,

$$\sum_{n=0}^{\infty} (n+1)\, a(n+1) + \left(\sum_{n=0}^{\infty} g(n) x^n\right)\left(\sum_{n=0}^{\infty} a(n) x^n\right) - \sum_{n=0}^{\infty} r(n) x^n = 0 \qquad (79)$$

Using Equ. (24), we have,

$$\sum_{n=0}^{\infty} (n+1)\, a(n+1) + \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a(k)\, g(n-k)\right] x^n - \sum_{n=0}^{\infty} r(n) x^n = 0, \qquad (80)$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+1)\, a(n+1) + \sum_{k=0}^{n} a(k)\, g(n-k) - r(n)\right] x^n = 0 .$$

Thus we have

$$a(n+1) = \frac{r(n) - \sum_{k=0}^{n} a(k)\, g(n-k)}{(n+1)}, \quad n = 0, 1, 2, \ldots \qquad (81)$$

Since $r(x) \neq 0$, we expect particular solution of Equ. (78), thus let

$$a(n) = a(0) \langle n, 0 \rangle + \langle n, 1 \rangle \qquad (82)$$

when $n = 0$, $a(0) = a(0) \langle 0, 0 \rangle + \langle 0, 1 \rangle$

we have $\langle 0, 0 \rangle = 1, \quad \langle 0, 1 \rangle = 0, \qquad (83)$

Since when $n = 0$ Equ. (81) should hold,

$$a(1) = r(0) - a(0)\, g(0)$$

$$\Leftrightarrow a(0) \langle 1, 0 \rangle + \langle 1, 1 \rangle = -a(0)\, g(0) + r(0)$$

thus we should have,

$$\langle 1, 0 \rangle = -g(0), \text{ and } \langle 1, 1 \rangle = r(0) \qquad (84).$$

Shifting indices in Equ. (81), we have

$$a(n) = \frac{r(n-1) - \sum_{k=0}^{n-1} a(k)\, g(n-k-1)}{n}, \quad n = 2, 3, 4 \ldots \tag{85}$$

$$= \frac{r(n-1)}{n} - \frac{1}{n}\left[\sum_{k=0}^{n-1} \left(a(0)\langle k,0\rangle + \langle k,1\rangle\right) g(n-k-1)\right]$$

$$= \frac{r(n-1)}{n} - \frac{1}{n}\left[\sum_{k=0}^{n-1} a(0)\langle k,0\rangle g(n-k-1)\right] - \frac{1}{n}\left[\sum_{k=0}^{n-1} \langle k,1\rangle g(n-k-1)\right]$$

$$= -\frac{a(0)}{n}\sum_{k=0}^{n-1} \langle k,0\rangle g(n-k-1) + \frac{r(n-1)}{n} - \frac{1}{n}\sum_{k=0}^{n-1} \langle k,1\rangle g(n-k-1)$$

$\Rightarrow \quad a(0)\langle n,0\rangle + \langle n,1\rangle$

$$= -\frac{a(0)}{n}\sum_{k=0}^{n-1} \langle k,0\rangle g(n-k-1) + \frac{r(n-1)}{n} - \frac{1}{n}\sum_{k=0}^{n-1} \langle k,1\rangle g(n-k-1),$$

thus we should have,

$$\langle n,0\rangle = -\frac{1}{n}\sum_{k=0}^{n-1} \langle k,0\rangle g(n-k-1), \quad n = 2, 3, 4 \ldots \tag{86}$$

with $\langle 0,0\rangle = 1$, and $\langle 1,0\rangle = -g(0)$,

and

$$\langle n,1\rangle = \frac{1}{n} r(n-1) - \frac{1}{n}\sum_{k=0}^{n-1} \langle k,1\rangle g(n-k-1), \quad n = 2, 3, 4 \ldots$$

with $\langle 0,1\rangle = 0$, and $\langle 1,1\rangle = r(0)$. (87)

Therefore general solution of Equ. (78) is

$$y = \sum_{n=0}^{\infty} a(n) x^n$$

$$= \sum_{n=0}^{\infty} \left(a(0)\langle n,0\rangle + \langle n,1\rangle\right) x^n$$

$$= \sum_{n=0}^{\infty} \left(a(0)\langle n,0\rangle x^n\right) + \sum_{n=0}^{\infty} \langle n,1\rangle x^n$$

$$= a(0)\sum_{n=0}^{\infty} \langle n,0\rangle x^n + \sum_{n=0}^{\infty} \langle n,1\rangle x^n \tag{88}$$

$$y = a(0)\left[\langle 0,0\rangle + \langle 1,0\rangle x + \sum_{n=2}^{\infty}\langle n,0\rangle x^n\right]$$

$$+\left[\langle 0,1\rangle + \langle 1,1\rangle x + \sum_{n=2}^{\infty}\langle n,1\rangle x^n\right]$$

$$= a(0)\left[1 - g(0)x + \sum_{n=2}^{\infty}\langle n,0\rangle x^n\right]$$

$$+\left[r(0)x + \sum_{n=2}^{\infty}\langle n,1\rangle x^n\right], \qquad (89)$$

where $\langle n,0\rangle = -\frac{1}{n}\sum_{k=0}^{n-1}\langle k,0\rangle g(n-k-1)$, with $n = 2,3,4\ldots$, (90)

and $\langle n,1\rangle = \frac{1}{n} r(n-1) - \frac{1}{n}\sum_{k=0}^{n-1}\langle k,1\rangle g(n-k-1)$, $n = 2,3,4\ldots$, (91)

with $\langle 0,0\rangle = 1$, $\langle 1,0\rangle = -g(0)$,

and $\langle 0,1\rangle = 0$, $\langle 1,1\rangle = r(0)$. (91)*

Now comparing Equ. (89) with Equ. (43)*, we can see that the first part of Equ. (89) - homogenious solution of Equ. (35) - exactly matches with the solution of Equ. (37) as it should be, and we have a particular solution - the second right-hand side of Equ. (89) as we must have. Thus we conclude that Equ. (89), with coefficients (90) and (91), is the general solution of Equ. (35).

Example) Solve $y' + e^x y = \frac{1}{\sqrt{1+x}}$, $x \neq -1$. (92)

By Equ. (8) $\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(1/2)}$, (93)

Since, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(n+1) = (n!)$, and $\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}(n!)}$,

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty}\frac{(-1)^n (2n)!}{2^{2n}(n!)^2}x^n \qquad (94)$$

Substituting Equ. (94) and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ in Equ. (92), we have,

$$y' + \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) y = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} x^n . \tag{95}$$

With $g(n) = \frac{1}{n!}$ and $r(n) = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$,

we have,

$$\left.\begin{aligned} &\langle 0.0\rangle = 1, \quad \langle 1.0\rangle = -g(0) = -1, \\ &\langle 0,1\rangle = 0, \quad \langle 1,1\rangle = r(0) = 1, \end{aligned}\right\} \tag{96}$$

and

$$\langle n.0\rangle = -\frac{1}{n}\sum_{k=0}^{n-1} \langle k,0\rangle \frac{1}{(n-k-1)!}, \quad n = 2,3,4\ldots \tag{97}$$

and $$\langle n,1\rangle = \frac{1}{n}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-2}((n-1)!)^2} - \frac{1}{n}\sum_{k=0}^{n-1}\langle k,1\rangle \frac{1}{(n-k-1)!}, \quad n=2,3,4\ldots, \tag{98}$$

$$y = a(0)\left[1 - x + \sum_{n=2}^{\infty} \langle n.0\rangle x^n\right] + \left[x + \sum_{n=2}^{\infty} \langle n,1\rangle x^n\right] . \tag{98}$$

Computer generation for $\langle n.0\rangle$ shows that,

$\langle 2.0\rangle = 0,$ $\quad\quad \langle 2,1\rangle = -0.75$

$\langle 3.0\rangle = 1.666666667E-01$ $\quad\quad \langle 3,1\rangle = 4.166666667E-02$

$\langle 4.0\rangle = 4.166666665E-02$ $\quad\quad \langle 4,1\rangle = -2.604166668E-02$

$\langle 5.0\rangle = -1.666666667E-02$ $\quad\quad \langle 5,1\rangle = 9.322916666E-02$

$\langle 6,0\rangle = -0.0125$ $\quad\quad \langle 6,1\rangle = -4.179687499E-02$

Thus $$y = a(0)\left[1 - x + 0.1\dot{6}x^3 + 0.041\dot{6}x^4 - 0.01\dot{6}x^5 + \ldots\right] + \left[x - 0.75x^2 + 0.041\dot{6}x^3 - 0.026041\dot{6}x^4 + \cdots\right] \tag{99}$$

Note here that the first part of right hand side of Equ. (99) exactly matches with Equ. (74) as it should be.

Computer programe for Equ. (95) is listed in the Appendix.

Here are some more power series operation — I derived them to use for Fourier Series in 1987, but they will be shown very useful in solving differential equations.

$$\sum_{n=0}^{\infty} f(n) \cos \pi n = \sum_{n=0}^{\infty} f(2n) - \sum_{n=0}^{\infty} f(2n+1) \quad \text{or,} \tag{100}$$

$$\sum_{n=1}^{\infty} f(n) \cos \pi n = \sum_{n=1}^{\infty} f(2n) - \sum_{n=1}^{\infty} f(2n-1) \tag{101}$$

Proof) Applying Equ. (1C) into Equ. (100), we have

$$\sum_{n=0}^{\infty} f(n) \cos \pi n = \sum_{n=0}^{\infty} f(2n) \cos(2\pi n) + \sum_{n=0}^{\infty} f(2n+1) \cos(2\pi n + \pi) \tag{102}$$

Since $\cos(2\pi n + \pi) = -1$, and $\cos(2\pi n) = 1$, for $n = 0, 1, 2 \ldots$

$$\sum_{n=0}^{\infty} f(n) \cos \pi n = \sum_{n=0}^{\infty} f(2n) - \sum_{n=0}^{\infty} f(2n+1).$$

Applying shifting indices to the left part of Equ. (101), and since $\cos(\pi n + \pi) = -\cos \pi n$, we have

$$\sum_{n=1}^{\infty} f(n) \cos \pi n = -\sum_{n=0}^{\infty} f(n+1) \cos \pi n \tag{103}$$

Applying Equ. (100) to the right part of Equ. (103),

$$\sum_{n=1}^{\infty} f(n) \cos \pi n = -\left[\sum_{n=0}^{\infty} f(2n+1) - \sum_{n=0}^{\infty} f(2n+2) \right]$$

$$= \sum_{n=0}^{\infty} f(2n+2) - \sum_{n=0}^{\infty} f(2n+1) \tag{104}$$

Applying shifting indices finally to Equ. (104), we have

$$\sum_{n=1}^{\infty} f(n)\cos \pi n = \sum_{n=1}^{\infty} f(2n) - \sum_{n=1}^{\infty} f(2n-1).$$

To save space, I state the followings without proof.

$$\sum_{n=0}^{\infty} f(n)\left[\cos \pi n + 1\right] = 2\sum_{n=0}^{\infty} f(2n) \qquad (105)$$

$$\text{or } \sum_{n=0}^{\infty} f(2n) = \frac{1}{2}\sum_{n=0}^{\infty} f(n)\left[\cos \pi n + 1\right] \qquad (105)^*$$

$$\sum_{n=1}^{\infty} f(n)\left[\cos \pi n + 1\right] = 2\sum_{n=1}^{\infty} f(2n) \qquad (106)$$

$$\text{or } \sum_{n=1}^{\infty} f(2n) = \frac{1}{2}\sum_{n=1}^{\infty} f(n)\left[\cos \pi n + 1\right] \qquad (106)^*$$

$$\sum_{n=0}^{\infty} f(n)\left[\cos \pi n - 1\right] = -2\sum_{n=0}^{\infty} f(2n+1) \qquad (107)$$

$$\text{or } \sum_{n=0}^{\infty} f(2n+1) = -\frac{1}{2}\sum_{n=0}^{\infty} f(n)\left[\cos \pi n - 1\right] \qquad (107)^*$$

$$\sum_{n=1}^{\infty} f(n)\left[\cos \pi n - 1\right] = -2\sum_{n=1}^{\infty} f(2n-1) \qquad (108)$$

$$= -2\sum_{n=0}^{\infty} f(2n+1)$$

$$\text{or } \sum_{n=1}^{\infty} f(2n-1) = \sum_{n=0}^{\infty} f(2n+1) = -\frac{1}{2}\sum_{n=1}^{\infty} f(n)\left[\cos \pi n - 1\right] \qquad (108)^*$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty} f(n)\left[\cos \pi n - 1\right]$$

$$\sum_{n=0}^{\infty} f(n)\left[1 - \cos \pi n\right] = 2\sum_{n=0}^{\infty} f(2n+1) \qquad (109)$$

$$\text{or } \sum_{n=0}^{\infty} f(2n+1) = \frac{1}{2}\sum_{n=0}^{\infty} f(n)\left[1 - \cos \pi n\right] \qquad (109)^*$$

$$\sum_{n=1}^{\infty} f(n)\left[1 - \cos \pi n\right] = 2\sum_{n=1}^{\infty} f(2n-1) \qquad (110)$$

$$= 2\sum_{n=0}^{\infty} f(2n+1)$$

$$\text{or } \sum_{n=1}^{\infty} f(2n-1) = \sum_{n=0}^{\infty} f(2n+1) = \frac{1}{2}\sum_{n=1}^{\infty} f(n)\left[1 - \cos \pi n\right] \qquad (110)^*.$$

$$\sum_{n=0}^{\infty} f(n) \sin \frac{\pi n}{2} = \sum_{n=0}^{\infty} (-1)^n f(2n+1) \qquad (111)$$

$$\sum_{n=0}^{\infty} f(n) \cos \frac{\pi n}{2} = \sum_{n=0}^{\infty} (-1)^n f(2n) \qquad (112)$$

$$\sum_{n=0}^{\infty} f(n) \sin \frac{3\pi n}{2} = \sum_{n=0}^{\infty} (-1)^{n+1} f(2n+1) \qquad (113)$$

$$\sum_{n=0}^{\infty} f(n) \cos \frac{3\pi n}{2} = \sum_{n=0}^{\infty} (-1)^n f(2n) \qquad (114)$$

There are many more useful formulas, but I think the above formulas are good enough to demonstrate solving differential equations.

Example) Solve $y' + \sin^2 x \, y = \frac{1}{\sqrt{1+x}}$, $x \neq -1$. (115)

At first glance, the above differential equation looks very difficult to obtain general solution, but as you will soon see, it's just a piece of cake.

By Equ. (7R), we have,

$$\sin^2 x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1}}{(2n)!} x^{2n}.$$

A little algebra gives us,

$$\sin^2 x = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n}}{(2n)!} x^{2n} \qquad (116)$$

From Equ. (112), we have,

$$\sum_{n=0}^{\infty} f(n) \cos \frac{\pi n}{2} = f(0) - \sum_{n=1}^{\infty} (-1)^{n+1} f(2n),$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} f(2n) = f(0) - \sum_{n=0}^{\infty} f(n) \cos\left(\frac{\pi n}{2}\right) \qquad (117)$$

Applying Equ. (117) to Equ. (116), we have,

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{2^n}{(n!)} \cos\left(\frac{\pi n}{2}\right) x^n. \qquad (118)$$

Thus, $g(0) = 0$, and $g(n) = -\frac{2^n}{2(n!)}\cos\left(\frac{\pi n}{2}\right)$. (119)

And by Equ. (94), i.e. $\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty}\frac{(-1)^n\,(2n)!}{2^{2n}\,(n!)^2}x^n$,
we have,

$$r(0) = 1, \text{ and } r(n) = \frac{(-1)^n\,(2n)!}{2^{2n}\,(n!)^2}. \quad (120)$$

Substituting Equ.(119) and Equ.(120) in Equ.(90), Equ.(91), and Equ.(91)* respectively, we have,

$$\left.\begin{array}{ll} \langle 0,0\rangle = 1, & \langle 1,0\rangle = -g(0) = 0; \\ \langle 0,1\rangle = 0, & \langle 1,1\rangle = r(0) = 1, \end{array}\right\} \quad (121)$$

and

$$\langle n,0\rangle = \frac{1}{n}\sum_{k=0}^{n-1}\langle k,0\rangle\,\frac{2^{n-k-1}}{2(n-k-1)!}\cos\left(\frac{\pi(n-k-1)}{2}\right), \quad n=2,3,4,\ldots \quad (122)$$

and

$$\langle n,1\rangle = \frac{1}{n}\,\frac{(-1)^{n-1}\,(2n-2)!}{2^{2n-2}\,((n-1)!)^2} + \frac{1}{n}\sum_{k=0}^{n-1}\langle k,1\rangle\,\frac{2^{n-k-1}}{2(n-k-1)!}\cos\left(\frac{\pi(n-k-1)}{2}\right), \quad n=2,3,4,\ldots, \quad (123)$$

thus

$$y = a(0)\left[1 + \sum_{n=2}^{\infty}\langle n,0\rangle x^n\right] + \left[x + \sum_{n=2}^{\infty}\langle n,1\rangle x^n\right]. \quad (124)$$

Computer generation for coefficients gives us,

$\langle 2,0\rangle = 0$, $\quad\langle 2,1\rangle = -0.25$

$\langle 3,0\rangle = -3.333333333E-01$ $\quad\langle 3,1\rangle = 0.125$

$\langle 4,0\rangle = 0$ $\quad\langle 4,1\rangle = -0.328125$

$\langle 5,0\rangle = 6.666666666E-02$ $\quad\langle 5,1\rangle = 0.1046875$

$\langle 6,0\rangle = 5.555555555E-02$ $\quad\langle 6,1\rangle = -6.293402783E-03$

$\vdots \qquad\qquad \vdots$

Therefore,

$$y = a(0)[1 - 0.\dot{3}x^3 + 0.0\dot{6}x^5 + 0.0\dot{5}x^6 + \cdots]$$

$$+ [x - 0.25x^2 + 0.125x^3 - 0.328125x^4 + \cdots] \quad (125)$$

Computer programs for Equ. (115) is listed in the Appendix.

Example) Solve $y' + \sin^2 x \, y = \sinh x$ (126)

We see, by Equ. (117R), that $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$. (127)

Applying Equ. (108) to Equ. (127), we have,

$$\sinh x = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} [1 - \cos \pi n] x^n. \quad (128)$$

Thus, as in Equ. (115),

$$g(0) = 0, \quad \text{and} \quad g(n) = -\frac{2^n}{2(n!)} \cos\left(\frac{\pi n}{2}\right), \quad (129)$$

and $r(0) = 0$, and $r(n) = \frac{[1 - \cos \pi n]}{2(n!)}$, (130)

$$\left.\begin{aligned} &\langle 0, 0\rangle = 1, \quad \langle 1, 0\rangle = -g(0) = 0 \\ &\langle 0, 1\rangle = 0, \quad \langle 1, 1\rangle = r(0) = 0 \end{aligned}\right\} \quad (131)$$

and

$$\langle n, 0\rangle = \frac{1}{n} \sum_{k=0}^{n-1} \langle k, 0\rangle \frac{2^{n-k-1}}{2(n-k-1)!} \cos\left(\frac{\pi(n-k-1)}{2}\right), \quad n = 2, 3, 4, \ldots \quad (132)$$

and $\langle n, 1\rangle = \frac{1}{n} \frac{[1 - \cos \pi(n-1)]}{2(n-1)!}$

$$+ \frac{1}{n} \sum_{k=0}^{n-1} \langle k, 1\rangle \frac{2^{n-k-1}}{2(n-k-1)!} \cos\left(\frac{\pi(n-k-1)}{2}\right), \quad n = 2, 3, 4 \ldots \quad (133)$$

Therefore we have,

$$y = a(0)\left[1 + \sum_{n=2}^{\infty} \langle n,0\rangle x^n\right] + \sum_{n=2}^{\infty} \langle n,1\rangle x^n \qquad (134)$$

Computer generated coefficients of Equ. (134) are

$\langle 2,0\rangle = 0$	$\langle 2,1\rangle = 0.5$
$\langle 3,0\rangle = -3.333333333E-01$	$\langle 3,1\rangle = 0$
$\langle 4,0\rangle = 0$	$\langle 4,1\rangle = 4.166666668E-02$
$\langle 5,0\rangle = 6.666666666E-02$	$\langle 5,1\rangle = -0.1$
$\langle 6,0\rangle = 5.555555555E-02$	$\langle 6,1\rangle = 1.388888889E-03$
$\vdots$	$\vdots$

Thus, $y = a(0)\left[1 - 0.\dot{3}x^3 + 0.0\dot{6}x^5 + 0.0\dot{5}x^6 + \ldots\right]$

$$+ \left[0.5x^2 + 0.041\dot{6}x^4 - 0.1x^5 + 0.0013\dot{8}x^6 \ldots\right] \qquad (135)$$

Comparision between Equ. (125) and Equ. (135) shows us the homogenious solution of Equ. (115) is identical to that of Equ. (126) as it must be.

So far, we solved only linear first order differential equations, then what about nonlinear differential equations? In case of nonlinear differential equations, the general solution is almost impossible but a few. I introduce here some new notations for later use.

Let $y = \sum_{n=0}^{\infty} a(n,1)x^n$, then we denote,

$$y' = \sum_{n=1}^{\infty} n\,a(n,1)x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1)\,a(n+1,1)x^n, \qquad (136)$$

and $y'' = \sum_{n=2}^{\infty} n(n-1)\,a(n,1)x^{n-2}$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, a(n+2,1)\, x^n, \tag{137}$$

$$y^2 = \sum_{n=0}^{\infty} a(n,2)\, x^n \tag{138}$$

$$\text{where} \quad a(n,2) = \sum_{m=0}^{n} a(m,1)\, a(n-m,1) \tag{139}$$

$$y^\alpha = \sum_{n=0}^{\infty} a(n,\alpha)\, x^n, \quad \alpha = 2, 3, 4, \dots \tag{140}$$

$$\text{where} \quad a(n,\alpha) = \sum_{m=0}^{n} a(m,\alpha-1)\, a(n-m,1). \tag{141}$$

Here I will prove only Equ. (139) and Equ. (141).

Proof) $$y^2 = \sum_{n=0}^{\infty} a(n,1) x^n \times \sum_{n=0}^{\infty} a(n,1) x^n \tag{142}$$

By Equ. (24),

$$y^2 = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} a(m,1)\, a(n-m,1) \right] x^n,$$

thus $$a(n,2) = \sum_{m=0}^{n} a(m,1)\, a(n-m,1).$$

$$y^\alpha = \sum_{n=0}^{\infty} a(n,\alpha)\, x^n$$

$$\Rightarrow \quad y^{\alpha-1} \cdot y = \sum_{n=0}^{\infty} a(n,\alpha-1) x^n \times \sum_{n=0}^{\infty} a(n,1) x^n$$

$$y^\alpha = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} a(m,\alpha-1)\, a(n-m,1) \right] x^n,$$

thus $$a(n,\alpha) = \sum_{m=0}^{n} a(m,\alpha-1)\, a(n-m,1).$$

Example) Mathematical Handbook of Formulas and Table by Murray R. Spiegel gives us at page 113, that if

$$y = c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots \tag{143}$$

then

$$x = C_1 y + C_2 y^2 + C_3 y^3 + C_4 y^4 + C_5 y^5 + C_6 y^6 + \dots \tag{144}$$

where

$$c_1 C_1 = 1 \tag{145}$$

$$c_1^3 C_2 = -c_2 \tag{146}$$

$$c_1^5 C_3 = 2c_2^2 - c_1 c_3 \tag{147}$$

$$c_1^7 C_4 = 5c_1 c_2 c_3 - 5c_2^3 - c_1^2 c_4 \tag{148}$$

$$c_1^9 C_5 = 6c_1^2 c_2 c_4 + 3c_1^2 c_3^2 - c_1^3 c_5 + 14c_2^4 - 21c_1 c_2^2 c_3 \tag{149}$$

$$c_1^{11} C_6 = 7c_1^3 c_2 c_5 + 84c_1 c_2^3 c_3 + 7c_1^3 c_3 c_4 - 28c_1^2 c_2 c_3^2 - c_1^4 c_6 - 28c_1^2 c_2^2 c_4 - 42c_2^5. \tag{150}$$

My unsatiable curiosity forced me to find out C_7, and if possible, C_n. So I tried and at last got the solution. I introduce this problem to make the readers get accustomed to Equ.(139) and Equ. (141).

For notational convenience, let $y = \sum_{n=1}^{\infty} a(n,1)x^n$, then

$$y^2 = \sum_{n=1}^{\infty} a(n,2)\, x^{n+1} \qquad a(n,2) = \sum_{k=1}^{n} a(k,1)\, a(n-k+1,1) \tag{151}$$

$$y^3 = \sum_{n=1}^{\infty} a(n,3)\, x^{n+2} \qquad a(n,3) = \sum_{k=1}^{n} a(k,2)\, a(n-k+1,1) \tag{152}$$

$$y^\alpha = \sum_{n=1}^{\infty} a(n,\alpha)\, x^{n+\alpha-1} \qquad a(n,\alpha) = \sum_{k=1}^{n} a(k,\alpha-1)\, a(n-k+1,1). \tag{153}$$

$$x = \sum_{\alpha=1}^{\infty} b(\alpha)\, y^\alpha \tag{154}$$

$$= b(1)y + b(2)y^2 + b(3)y^3 + b(4)y^4 + \cdots$$

$$= b(1)a(1,1)x + b(1)a(2,1)x^2 + b(1)a(3,1)x^3 + b(1)a(4,1)x^4 + \cdots$$
$$b(2)a(1,2)x^2 + b(2)a(2,2)x^3 + b(2)a(3,2)x^4 + \cdots$$
$$+ b(3)a(1,3)x^3 + b(3)a(2,3)x^4 + \cdots$$
$$b(4)a(1,4)x^4 + \cdots$$

$$= \left(\sum_{\alpha=1}^{1} b(\alpha)\, a(1-\alpha+1,\alpha)\right)x + \left(\sum_{\alpha=1}^{2} b(\alpha)\, a(2-\alpha+1,\alpha)\right)x^2$$
$$+ \left(\sum_{\alpha=1}^{3} b(\alpha)\, a(3-\alpha+1,\alpha)\right)x^3 + \cdots$$

$$x = \sum_{n=1}^{\infty}\left[\sum_{\alpha=1}^{n} b(\alpha)a(n-\alpha+1,\alpha)\right]x^n \qquad (155)$$

$$\therefore \quad b(1)a(1,1) = 1 \qquad (156)^*$$

$$\sum_{\alpha=1}^{n} b(\alpha)\, a(n-\alpha+1,\alpha) = 0, \text{ for all } n = 2,3,4,\ldots$$

$$\Rightarrow b(n)a(1,n) + \sum_{\alpha=1}^{n-1} b(\alpha)a(n-\alpha+1,\alpha) = 0$$

Therefore,

$$b(n) = -\frac{\sum_{\alpha=1}^{n-1} b(\alpha)\, a(n-\alpha+1,\alpha)}{a(1,n)}, \text{ for all } n = 2,3,4\ldots \qquad (156)$$

where

$$a(n,\alpha) = \sum_{k=1}^{n} a(k,\alpha-1)\, a(n-k+1,1). \qquad (157)$$

You may ask what is the use of Equ. (156) and Equ. (157)? Now see where they can be useful!

Exp) Let $y = \sin(x)$, then by Equ. (IM), we have,

$$y = \sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}. \qquad (IM)$$

Applying Equ. (111) to Equ. (IM), we have,

$$y = \sin(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\sin\frac{\pi n}{2}x^n. \qquad (157)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\sin\frac{\pi n}{2}x^n \qquad (158)$$

Let $a(n,1) = \frac{1}{n!}\sin\frac{\pi n}{2}$, then I want to express x in terms of y, i.e. $x = \sum_{n=1}^{\infty} b(n)y^n$. (159)

Then by Equ. $(156)^*$, (156), and (157), with $a(n) = \frac{1}{n!}\sin\frac{\pi n}{2}$, we have,

$$b(1)\cdot 1 = 1 \qquad (160)$$

$$b(n) = -\frac{\sum_{\alpha=1}^{n-1} b(\alpha)\, a(n-\alpha+1, \alpha)}{a(1, n)}, \text{ for all } n = 2, 3, 4, \ldots, \quad (161)$$

where
$$a(n, \alpha) = \sum_{k=1}^{n} a(k, \alpha-1) \cdot \frac{1}{(n-k+1)!} \sin \frac{\pi(n-k+1)}{2} \quad (162)$$

$$n = 2, 3, 4, \quad \alpha = 2, 3, 4 \ldots$$

and
$$a(n, 1) = \frac{1}{n!} \sin \frac{\pi n}{2}. \quad (163)$$

At this point, you may wonder how can we evaluate $b(n)$. The procedure to calculate $b(n)$ is very simple. Since $a(n,1)$ is known for all $n \in N$, we first calculate $a(n, \alpha)$ using $a(n,1)$, then using Equ. (161), we calculate $b(n)$.

$$x \cdot \sum_{n=0}^{\infty} b\langle n\rangle x^{n+2r-4} - y^3 = 0.$$

$$y^2 = x^r \sum_{n=0}^{\infty} a\langle n,1\rangle x^n \times x^r \sum_{n=0}^{\infty} a\langle n,1\rangle x^n$$

$$= x^{2r} \sum_{n=0}^{\infty} a\langle n,2\rangle x^n$$

$$a\langle n,2\rangle = \sum_{k=0}^{n} a\langle k,1\rangle a\langle n-k,1\rangle$$

$$y^3 = x^{2r} \sum_{n=0}^{\infty} a\langle n,2\rangle x^n \times x^r \sum_{n=0}^{\infty} a\langle n,1\rangle x^n.$$

$$y^3 = x^{3r} \sum_{n=0}^{\infty} a\langle n,3\rangle x^n.$$

$$a\langle n,3\rangle = \sum_{k=0}^{n} a\langle k,2\rangle a\langle n-k,1\rangle$$

$$= \sum_{n=0}^{\infty} a\langle n,3\rangle x^{n+3r}.$$

$$\sum_{n=0}^{\infty} b\langle n\rangle x^{n+2r-3} - \sum_{n=0}^{\infty} a\langle n,3\rangle x^{n+3r}. = 0.$$

$$\boxed{\sum_{n=-r-3}^{\infty} b\langle n+r+3\rangle x^{n+3r} - \sum_{n=0}^{\infty} a\langle n,3\rangle x^{n+3r}} = 0$$

$$x^{2r-3} \sum_{n=0}^{\infty} b\langle n\rangle x^n - x^{3r} \sum_{n=0}^{\infty} a\langle n,3\rangle x^n = 0.$$

$$\sum_{n=(r+3)}^{\infty} b\langle n-r-3\rangle x^{n+3r}$$

$-r-3+3r$

$$\sum_{n=-3}^{\infty} b\langle n+3\rangle x^{n+2r} - \sum_{n=0}^{\infty} a\langle n,3\rangle x^{n+3r} = 0.$$

-2

-1

$2r-3$

$\boxed{r+3}+$

$n = \cancel{0}(r+3)$

$$b\langle 0\rangle x^{2r-3} + b\langle 1\rangle x^{2r-2} + b\langle 2\rangle x^{2r-1}$$

-3 -2 -1

$(\cancel{n}+z)+2r-3$

$= \cancel{n}+3r$

$\boxed{r은정수}$

$$+ \sum_{n=0}^{\infty} b\langle n+3\rangle x^{n+2r} - \sum_{n=0}^{\infty} a\langle n,3\rangle x^{n+3r} = 0.$$

$\boxed{z = r+3}$

$$- x^r \sum_{n=0}^{\infty} a\langle n,3\rangle x^{n+2r} = 0.$$

$z = \boxed{r+3}$

$(\cancel{n}+z)+2r-3 = \cancel{n}+3r$

$z+2r-3 = 2r-3$

$z+2r-3 = 3r$

$\cancel{n}+z+2r-3 = \cancel{n}+2r-3$

$z = 3r-2r+3$

$$x^{\frac{1}{2}} \frac{d^2y}{dx^2} = y^{\frac{3}{2}}$$

$\cancel{y = 144x^{-3}}$

$$\left(\begin{matrix} x=0 \text{ then } y=1 \\ x \to \infty \quad y \to 0 \end{matrix} \right) \quad y'(0) =$$

$$x(y'')^2 = y^3$$

$x($

$y = \sum_{n=0}^{\infty} a\langle n \rangle x^{n+r}$

$y' = \sum_{n=0}^{\infty} a\langle n \rangle (n+r) x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} a\langle n \rangle (n+r)(n+r-1) x^{n+r-2}$

$(y'')^2 = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a\langle n-k \rangle (k+r)(k+r-1) \right.$

$a\langle n.$

$y = \sum_{n=0}^{\infty} a\langle n.1 \rangle x^{n+r}$

$y' = \sum_{n=0}^{\infty} a\langle n.1 \rangle (n+r) x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} a\langle n.1 \rangle (n+r)(n+r-1) x^{n+r-2}$.

$a\langle 0.1 \rangle x^{n+r} = a\langle 0.1 \rangle x^{r}$

x^n

$(y'')^2 = \sum_{n=0}^{\infty} a\langle n.1 \rangle (n+r)(n+r-1) x^{n+r-2} \times \sum_{n=0}^{\infty} a\langle n,1 \rangle (n+r)(n+r-1) x^{n+r-2}$

$= x^{-4} \sum$

$= x^{2r-4} \sum_{n=0}^{\infty} a\langle n.1 \rangle (n+r)(n+r-1) x^n \sum_{n=0}^{\infty} a\langle n.1 \rangle (n+r)(n+r-1) x^n$

$= x^{2r-4} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a\langle k.1 \rangle (k+r)(k+r-1)\ a\langle n-k.1 \rangle (n-k+r)(n-k+r-1) \right] x^n$

$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a\langle k.1 \rangle (\right.$

$= \sum_{n=0}^{\infty} b\langle n \rangle x^{n+2r-4}$.

$b\langle n \rangle = \sum_{k=0}^{n} a\langle k.1 \rangle (k+r)(k+r-1)\ a\langle n-k.1 \rangle (n-k+r)(n-k+r-1)$