$$\int \underline{\sin^m(ax)}\ \underline{\cos^n(bx)}\,dx$$

$$y_1 = \cos(ax)\sin^{m-1}(ax)\cos^n(bx)$$

$$y_2 = \sin^m(ax)\sin(bx)\cos^{n-1}(bx)$$

If $n^2b^2 - m^2a^2 \neq 0$,

$$\int \sin^m(ax)\cos^n(bx)\,dx$$

$$= \frac{1}{n^2b^2 - m^2a^2}\left\{ ma\cos(ax)\sin^{m-1}(ax)\cos^n(bx) + nb\sin^m(ax)\sin(bx)\cos^{n-1}(bx)\right\}$$

$$- \frac{m(m-1)a^2}{n^2b^2 - m^2a^2}\int \sin^{m-2}(ax)\cos^n(bx)\,dx$$

$$+ \frac{n(n-1)b^2}{n^2b^2 - m^2a^2}\int \sin^m(ax)\cos^{n-2}(bx)\,dx$$

$$y_1 = \cos(ax)\sin^{m-1}(ax)\cos^n(bx)$$

$$\begin{aligned} y_1' = & -a\sin(ax)\sin^{m-1}(ax)\cos^n(bx) \\ & +(m-1)a\cos^2(ax)\sin^{m-2}(ax)\cos^n(bx) \\ & -nb\cos(ax)\sin^{m-1}(ax)\sin(bx)\cos^{n-1}(bx) \end{aligned}$$

$$\begin{aligned} y_1' = & -a\sin^m(ax)\cos^n(bx) \\ & +(m-1)a\left\{1-\sin^2(ax)\right\}\sin^{m-2}(ax)\cos^n(bx) \\ & -nb\cos(ax)\sin^{m-1}(ax)\sin(bx)\cos^{n-1}(bx) \end{aligned}$$

$$\begin{aligned} y_1' = & -ma\sin^m(ax)\cos^n(bx) \\ & +(m-1)a\sin^{m-2}(ax)\cos^n(bx) \\ & -nb\cos(ax)\sin^{m-1}(ax)\sin(bx)\cos^{n-1}(bx) \end{aligned}$$

$$\int y_1'\,dx = \int \boxed{\qquad\qquad}\,dx$$

$$\boxed{\begin{aligned} & \cos(ax)\sin^{m-1}(ax)\cos^n(bx) \\ & = -ma\int\sin^m(ax)\cos^n(bx)\,dx \quad \cdots (A) \\ & \quad +(m-1)a\int\sin^{m-2}(ax)\cos^n(bx)\,dx \\ & \quad -\underbrace{nb\int\cos(ax)\sin^{m-1}(ax)\sin(bx)\cos^{n-1}(bx)\,dx}_{(C)} \end{aligned}}$$

$$y_2 = \sin^m(ax)\sin(bx)\cos^{n-1}(bx)$$

$$\begin{aligned} y_2' &= m\cdot a\cdot \cos(ax)\sin^{m-1}(ax)\sin(bx)\cos^{n-1}(bx) \\ &\quad + b\sin^m(ax)\cos(bx)\cos^{n-1}(bx) \\ &\quad - (n-1)b\sin^m(ax)\sin^2(bx)\cos^{n-2}(bx) \end{aligned}$$

$$\begin{aligned} y_2' &= ma\cos(ax)\sin^{m-1}(ax)\sin(bx)\cos^{n-1}(bx) \\ &\quad + b\sin^m(ax)\cos^n(bx) \\ &\quad - (n-1)b\sin^m(ax)\left\{1-\cos^2(bx)\right\}\cos^{n-2}(bx) \end{aligned}$$

$$\begin{aligned} y_2' &= ma\cos(ax)\sin^{m-1}(ax)\sin(bx)\cos^{n-1}(bx) \\ &\quad + nb\sin^m(ax)\cos^n(bx) \\ &\quad - (n-1)b\sin^m(ax)\cos^{n-2}(bx) \end{aligned}$$

$$\int y_2'\,dx = \int \boxed{}\,dx$$

$$(B)\ldots\quad \boxed{\begin{aligned} &\sin^m(ax)\sin(bx)\cos^{n-1}(bx) \\ &= \underbrace{ma\int \cos(ax)\sin^{m-1}(ax)\sin(bx)\cos^{n-1}(bx)\,dx}_{(C)} \\ &\quad + nb\int \sin^m(ax)\cos^n(bx)\,dx \\ &\quad - (n-1)b\int \sin^m(ax)\cos^{n-2}(bx)\,dx \end{aligned}}$$

(A) × ma + (B) × nb

then (C) is eliminated.

$$ma\cos(ax)\sin^{m-1}(ax)\cos^n(bx)$$
$$+\,nb\sin^m(ax)\sin(bx)\cos^{n-1}(bx)$$
$$=(-m^2a^2+n^2b^2)\int\sin^m(ax)\cos^n(bx)\,dx$$
$$m(m-1)a^2\int\sin^{m-2}(ax)\cos^n(bx)\,dx$$
$$-\,n(n-1)b^2\int\sin^m(ax)\cos^{n-2}(bx)\,dx$$

If $-m^2a^2+n^2b^2\neq 0$,

$$\int\sin^m(ax)\cos^n(bx)\,dx$$
$$=\frac{1}{-m^2a^2+n^2b^2}\left\{ma\cos(ax)\sin^{m-1}(ax)\cos^n(bx)+nb\sin^m(ax)\sin(bx)\cos^{n-1}(bx)\right\}$$
$$-\frac{m(m-1)a^2}{-m^2a^2+n^2b^2}\int\sin^{m-2}(ax)\cos^n(bx)\,dx$$
$$+\frac{n(n-1)b^2}{-m^2a^2+n^2b^2}\int\sin^m(ax)\cos^{n-2}(bx)\,dx$$

$\int \cos^n(ax)\sin(bx)\,dx$

$y_1 = \sin(ax)\cos^{n-1}(ax)\sin(bx)$

$y_2 = \cos^n(ax)\cos(bx)$

$$\boxed{\begin{aligned}\int \cos^n(ax)\sin(bx)\,dx &= \frac{1}{n^2a^2-b^2}\Big\{na\sin(ax)\cos^{n-1}(ax)\sin(bx) \\ &\quad + b\cos^n(ax)\cos(bx)\Big\} \\ &\quad + \frac{n(n-1)a^2}{n^2a^2-b^2}\int \cos^{n-2}(ax)\sin(bx)\,dx\end{aligned}}$$

$\cos^3(2x)\sin(6x)$

$$y_1 = \underline{\sin(ax)}\ \underline{\cos^{n-1}(ax)}\ \sin(bx)$$

$$y_1' = +a\cos^n(ax)\sin(bx)$$

$$-\underline{(n-1)a}\ \sin^2(ax)\cos^{n-2}(ax)\sin(bx)$$

$$(1-\cos^2(ax))$$

$$b\sin(ax)\cos^{n-1}(ax)\cos(bx)$$

$$= \underline{a\cos^n(ax)\sin(bx)}\ \times$$

$$-(n-1)a\cos^{n-2}(ax)\sin(bx)$$

$$\underline{(n-1)a\cos^n(ax)\sin(bx)}\ \times$$

$a + na - a$

$$b\sin(ax)\cos^{n-1}(ax)\cos(bx)$$

$$= na\cos^n(ax)\sin(bx)$$

$$-(n-1)a\cos^{n-2}(ax)\sin(bx)$$

$$b\sin(ax)\cos^{n-1}(ax)\cos(bx)$$

$$\boxed{\begin{aligned}&\sin(ax)\cos^{n-1}(ax)\sin(bx)\\ &= na\int\cos^n(ax)\sin(bx)\,dx\\ &-(n-1)a\int\cos^{n-2}(ax)\sin(bx)\,dx\\ &\quad b\int\sin(ax)\cos^{n-1}(ax)\cos(bx)\,dx\end{aligned}}$$

(A)

$y_2 = \cos^n(ax)\cos(bx)$

$y_2' = -na\sin(ax)\cos^{n-1}(ax)\cos(bx)$

$-b\cos^n(ax)\sin(bx)$

$$\boxed{\begin{aligned}&\cos^n(ax)\cos(bx)\\ &= -na\int \sin(ax)\cos^{n-1}(ax)\cos(bx)\,dx\\ &\quad -b\int \cos^n(ax)\sin(bx)\,dx\end{aligned}}$$

(B)

(~~A~~)

nab

$-nab$

(A) $\times\, \underline{na}$ + $\underline{(B)\, b}$ (4)

$na \sin(ax) \cos^{n-1}(ax) \sin(bx)$

$b \cos^n(ax) \cos(bx)$

$$= n^2a^2 \int \cos^n(ax) \sin(bx)\,dx$$

$$-b^2 \int \underline{\cos^n(ax) \sin(bx)}\,dx$$

$$-n(n-1)a^2 \int \cos^{n-2}(ax) \sin(bx)\,dx$$

$\Rightarrow$

$$(n^2a^2-b^2) \int \underline{\cos^n(ax) \sin(bx)\,dx}$$

$$-n(n-1)a^2 \int \cos^{n-2}(ax) \sin(bx)\,dx$$

$$= \{na \underline{\sin(ax) \cos^{n-1}(ax) \sin(bx)}$$

$$+ b \cos^n(ax) \cos(bx)\}$$

$$\boxed{\begin{aligned} &\int \cos^n(ax) \sin(bx)\,dx \\ &= \frac{1}{n^2a^2-b^2}\{na \sin(ax) \cos^{n-1}(ax) \sin(bx) \\ &\qquad + b \cos^n(ax) \cos(bx)\} \\ &+ \frac{n(n-1)a^2}{n^2a^2-b^2} \int \cos^{n-2}(ax) \sin(bx)\,dx \end{aligned}}$$

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$$\int \sin^n(ax)\cos(bx)\,dx$$

$$\begin{cases} y_1 = \cos(ax)\sin^{n-1}(ax)\cos(bx) \\ y_2 = \sin^n(ax)\sin(bx) \end{cases}$$

$$\boxed{\begin{aligned} &\int \sin^n(ax)\cos(bx)\,dx \\ &= \frac{1}{b^2-n^2a^2}\Big\{ na\cos(ax)\sin^{n-1}(ax)\cos(bx) \\ &\qquad + b\sin^n(ax)\sin(bx)\Big\} \\ &\quad - \frac{n(n-1)a^2}{b^2-n^2a^2}\int \sin^{n-2}(ax)\cos(bx)\,dx. \end{aligned}}$$

$y_1 = \cos(ax)\sin^{n-1}(ax)\cos(bx)$

$y_1' = -a\sin^n(ax)\cos(bx)$

$(n-1)a\cos^2(ax)\sin^{n-2}(ax)\cos(bx)$

$(1-\sin^2(ax))$

$-b\cos(ax)\sin^{n-1}(ax)\sin(bx)$

$= \cancel{-a\sin^n(ax)\cos(bx)}$

$(-a)-(na-a)$

$= \cancel{-a} - na \cancel{+a}$

$(n-1)a\sin^{n-2}(ax)\cos(bx)$

$\cancel{-(n-1)a\sin^n(ax)\cos(bx)}$

$-b\cos(ax)\sin^{n-1}(ax)\sin(bx)$

$= -na\sin^n(ax)\cos(bx)$

$(n-1)a.\sin^{n-2}(ax)\cos(bx)$

$-b\cos(ax)\sin^{n-1}(ax)\sin(bx)$

$\cos(ax)\sin^{n-1}(ax)\cos(bx)$

$= -na\int \sin^n(ax)\cos(bx)\,dx$ $\times$

$+(n-1)a\int \sin^{n-2}(ax)\cos(bx)\,dx$ $\times$

$-b\int \cos(ax)\sin^{n-1}(ax)\sin(bx)\,dx$ $\times$ ...(A)

$$y_2 = \sin^n(ax)\sin(bx)$$

$$y_2' = na\cos(ax)\sin^{n-1}(ax)\sin(bx)$$

$$b\sin^n(ax)\cos(bx)$$

$$\sin^n(ax)\sin(bx) = na\int \cos(ax)\sin^{n-1}(ax)\sin(bx)\,dx + b\int \sin^n(ax)\cos(bx)\,dx \quad \ldots(B)$$

$(A) \times na + (B)\, b.$

$$na\cos(ax)\sin^{n-1}(ax)\cos(bx)$$

$$+\, b\sin^{n}(ax)\sin(bx)$$

$$= -n^2a^2 \int \sin^{n}(ax)\cos(bx)\,dx \quad (A)$$

$$b^2 \int \sin^{n}(ax)\cos(bx)\,dx \quad (B)$$

$$n(n-1)a^2 \int \sin^{n-2}(ax)\cos(bx)\,dx$$

$$= (b^2 - n^2a^2) \int \sin^{n}(ax)\cos(bx)\,dx.$$

$$+\, n(n-1)a^2 \int \sin^{n-2}(ax)\cos(bx)\,dx.$$

$$(b^2 - n^2a^2) \int \sin^{n}(ax)\cos(bx)\,dx.$$

$$= \left[na\cos(ax)\sin^{n-1}(ax)\cos(bx) + b\sin^{n}(ax)\sin(bx)\right]$$

$$-\, n(n-1)a^2 \int \sin^{n-2}(ax)\cos(bx)\,dx.$$

$$\boxed{\int \sin^{n}(ax)\cos(bx)\,dx = \frac{1}{(b^2-n^2a^2)}\left[na\cos(ax)\sin^{n-1}(ax)\cos(bx) + b\sin^{n}(ax)\sin(bx)\right] - \frac{n(n-1)a^2}{b^2-n^2a^2}\int \sin^{n-2}(ax)\cos(bx)\,dx.}$$